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O.R. Applications

Positive dynamic systems with entropy operator: Application to labour market modelling

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Abstract

Nonlinear dynamic systems with positive solutions and the parametric problem of entropy maximization (entropy operator) are considered. The continuity, differentiability and boundedness of the entropy operator and the boundedness of the solutions of the dynamic system are derived using the global implicit function theorem. The technique of positive dynamic systems is applied to modelling of the labour market dynamics. The model is based on description of cohorts competition and of labour demand–supply interaction. A modification of the random search is used for parametric identification of the model. The model is tested on real data from the EU-countries.

Keywords: Dynamic system; Positive solution; Entropy maximization; Entropy operator; Boundedness of solution; Random search; Labour market modelling; Cohorts competition

1. Introduction

The book by Wilson (1991) and the paper by Popkov and Ryazantsev (1980) appeared almost simultaneously in 1980–1981. The concept of a dynamic system with entropy operator (DSEO-concept) was proposed in these publications for the first time. Certainly it was meant for specific dynamic problems of facilities location and spatial population distribution.

The main idea of these works consists of three parts. One of them is a separation of the general process into two processes—slow and fast. The second part is based on an assumption that the slow process is deterministic and the fast process is stochastic. The third part is the presentation of the fast stochastic process as a sequence of local-stationary states. If the fast process is a stochastic distributive process then its local-stationary state can be described as a state with maximum entropy over some feasible set.

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The DSEO-concept is useful for mathematical modelling of the resources exchange in regional systems (Popkov et al., 1998), spatially-temporary evolution of the biological communities (Volterra, 1931), and some processes of the chemical kinetics with a common catalyst (Eigen and Schuster, 1979).

The DSEO contains the nonlinearity which is described by the parametric mathematical programming problem with entropy as a goal function. In the general case the entropy operator is characterized by a multivalued mapping. Till now only some simplest types of DSEO have been investigated, namely DSEO for which the entropy operator is described by the single-valued regular mapping (Popkov and Shvetsov, 1990; Shvetsov, 1989; Kitaev, 1997; Bobilev and Popkov, 2002).

In this paper the positive DSEO is considered. It is a rather widespread subclass of the DSEO, which is useful for applications. Our contribution to the study of such systems is the development of the mathematical methods of qualitative analysis of the entropy operator (continuity, smoothness, boundedness) and of the positive dynamic systems with the entropy operator (boundedness of solutions).

The second part of the paper is devoted to labour market modelling. Below it will be shown that the labour market model belongs to the class of positive DSEO; it is a good application of the theoretical results.

We also propose an algorithm for the model identification and represent the numerical study of this model based on real data from the EU-countries.

2. Description of positive DSEO

Consider a dynamic system with two internal processes: slow $x(t) = \{x_1(t), \dots, x_n(t)\}$ and fast $y(t) = \{y_1(t), \dots, y_m(t)\}$. The slow process is deterministic and the fast is stochastic. We assume that the time relaxation τ_{fast} of the fast process is considerably less than the time relaxation τ_{slow} of the slow process. So the fast process can be considered as a sequence of the local-stationary states $y^*(t)$ characteristics of which depend on the slow process, i.e. $y^*(t) = y^*(x(t))$. In particular, this assumption is realized in the macrosystems with self-reproduction and resources distribution (see Popkov, 1989, 1993).

Local-stationary state $y^*(t) = \{y_1^*(t), \dots, y_m^*(t)\}$ (where t is fixed) can be modelled like a random distribution of the virtual elements among m boxes, capacities of which $S_1(x(t)), \dots, S_m(x(t))$ are finite. The elements can occupy the jth box with prior probability $w_j(x(t))$ and independently from the others. Distribution of the elements among boxes is accompanied by consumption of $(r + l)$ types of the resources. Denote the consumption function for the kth resource as $\phi_k(y, x)$, where $k = 1, \dots, (r + l)$.

The generalized information entropy by Boltzmann (BE) (Popkov, 1996) is a characteristic of the random distribution $y(t)$:

$$H_{\text{B}}(y, x) = -\sum_{j=1}^{m} y_j \ln \frac{y_j}{e w_j(x)}, \tag{1}$$

where $e \cong 2.718$, $y \in R_+^m = \{y_j : y_j \geqslant 0, j = 1, \dots, m\}$ and $0 \ln 0 = 0$.

If we can introduce a cell-structure in the boxes (for example, the capacity S_j is the number of cells in the jth box), we use the generalized information entropy by Fermi (FE):

$$H_{\text{F}}(y, x) = -\sum_{j=1}^{m} \left(y_j \ln \frac{y_j}{w_j(x)} + (S_j - y_j) \ln(S_j - y_j) \right), \tag{2}$$

where $0 \leqslant y_j \leqslant S_j(x)$, $j = 1, \dots, m$, and $0 \ln 0 = 0$.

And by Einstein (EE):

$$H_{\mathrm{E}}(y,x) = -\sum_{j=1}^{m}\left(y_j \ln\frac{y_j}{w_j(x)} - (S_j + y_j)\ln(S_j + y_j)\right), \tag{3}$$

where $y \in R_+^m$ and $0 \ln 0 = 0$.

When the stocks of $(r + l)$ resources are limited then the distributions $y(t)$ must comply with certain constraints which describe the feasible set

$$D(x) = \{y : \phi_k(y,x) = 0,\ \phi_{r+s}(y,x) \leqslant 0;\ k = 1,\dots,r,\ s = 1,\dots,l\}. \tag{4}$$

Thus, using the classical variation principle of statistical physics (Landau and Lifschitz, 1964) the local-stationary state of the fast process can be represented in the following form:

$$y^*(x) = \arg\max_y (H(y,x)|y \in D(x)) > 0. \tag{5}$$

This expression describes the map $R^n \mapsto R^m$, an operator of which is called the *entropy operator*.

Now consider the slow process of the system. As the local-stationary state of the fast process is determined by the state of the slow process, then we can characterize the state of the system by the vector $x(t)$.

A dynamic system is called positive if $x(t) \geqslant 0$ for any initial states $x(0) > 0$. This property is always valid for the mathematical model of the system in the rate terms:

$$\frac{1}{x} \otimes \frac{\mathrm{d}x}{\mathrm{d}t} = g(x, y^*(x)), \tag{6}$$

where $\otimes$ means the coordinate-wise operation.

In this equation g is a vector-function ($g \in R^n$) such that $g(0,0) \neq \infty$. In fact, we can see that $x(t) \geqslant 0$ for all $t \geqslant 0$, if $x(0) > 0$, i.e. Eq. (6) describes the positive dynamic system. Further, we will consider the function $g(x, y^*)$ of the following form:

$$g(x,y^*) = a - x \otimes Q(x,y^*), \qquad a, Q(x,y^*) \in R_+^n. \tag{7}$$

Thus, the system is described by the following equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x \otimes (a - x \otimes Q(x, y^*(x))), \tag{8}$$

where $y^*(x)$ is the entropy operator (5).

We can see that this equation transforms to the Lotka–Volterra (Volterra, 1931) equation if $Q(x) = Q$, where Q is a constant matrix.

The function $Q(x, y^*)$ (8) can take different forms. One of them that we will use here, arises as a result of the projection $y^*(x)$ on R^n ($m > n$). Denoting projector as $P = [p_{ij} \geqslant 0, i = 1,\dots,n;\ j = 1,\dots,m]$, we obtain

$$Q(x) = Py^*(x). \tag{9}$$

Thus, the class of the dynamic systems which will be studied in this article is described by the following equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x \otimes (a - Py^*(x) \otimes x), \tag{10}$$

where $y^*(x)$ is the entropy operator (5).

3. Entropy operator

Consider the entropy operator (5) with the feasible set (4) described by the system of equalities:

$$D(x) = \{y : \phi_k(y,x) = 0,\ k = 1,\dots,r\}. \tag{11}$$

Call the operator (5) and (11) *the conditionally optimal entropy operator*.

A general approach for studying of the entropy operator is based on the Lagrange method. We will show this approach for the BE-operator and make a few remarks on the FE- and EE-operators.

Consider a compact subset X in R^n and point $x \in X$. Introduce the Lagrange function

$$L(y, x, \lambda) = H_{\mathrm{B}}(y, x) - \sum_{k=1}^{r} \lambda_k \phi_k(y, x). \tag{12}$$

Optimality conditions for the BE-operator can be represented in the following form:

$$\begin{aligned} \psi_j(y, x, \lambda) &= \ln \frac{w_j(x)}{y_j} - \sum_{k=1}^{r} \lambda_k \frac{\partial \phi_k(y, x)}{\partial y_j} = 0, \quad j = 1, \dots, m, \\ \phi_k(y, x) &= 0, \quad k = 1, \dots, r. \end{aligned} \tag{13}$$

Theorem 1. *Let the following conditions be valid for all points $x \in X$:*

(a) *the functions $\phi_k(y, x)(k = 1, \dots, r)$ are twice continuously differentiable and monotonically increasing over y;*
(b) *the Hessian $[\frac{\partial^2 \phi_k(y,x)}{\partial y_j \partial y_i}]$ is a positive definite matrix;*
(c) *there exists subset $M(x) \subset R^m$ where the Jacobian $[\frac{\partial \phi_k(y,x)}{\partial y_j}]$ has a full rank equal r;*
(d) *for all $j = 1, \dots, m$*

$$\phi_k(0, x) < 0, \quad \phi_k(y_1, \dots, y_{j-1}, w_j(x), y_{j+1}, \dots, y_m; x) > 0, \qquad k = 1, \dots, r.$$

Then the Hessian of the Lagrange function (12) *is a negative definite matrix.*

Similar results can be obtained for the FE- and EE-operators, if we introduce the following additional conditions (separability of the functions ϕ_k):

$$\phi_k(y, x) = \sum_{l} \phi_{kl}(y, x), \quad k = 1, \dots, r. \tag{14}$$

The general idea of the proof is based on the investigation of the quadratic form $G(y, x, \lambda) = \langle L_{yy}(y, x, \lambda)h, h \rangle$, where L_{yy} is the Hessian of the Lagrange function and the vector $h \in R^m$. Key is the proof of nonnegativeness of the Lagrange multipliers, where the conditions of Theorem 1 are used. The complete proof is given in Appendix A.

The qualitative properties of the BE-operator are investigated using the "global" theorem of implicit functions (see Theorem A.1).

Theorem 2. *Let the conditions of Theorem 1 be valid and assume the functions $w_1(x), \dots, w_m(x)$ are continuous. Then for all $x \in X$ there exists the continuous BE-operator $y^*(x) \geqslant 0$.*

Theorem 3. *Let the conditions of Theorem 1 be valid and assume the functions $w(x)$ and $\phi(y, x)$ have the smoothness's order over $x \in X$ equal p. Then for all $x \in X$ the BE-operator has the smoothness's order equal p.*

The proofs can be found in Popkov (1995).

Lemma 1. *Let the conditions of Theorem 1 be valid. Then the BE-operator is bounded, i.e. there exists a positive constant C such that $y^*(x) \leqslant C$ (for the FE-operator $C = \max(S_1, \dots, S_m)$).*

The proof of Lemma 1 is given in Appendix A.

4. Boundedness of the DSEO solutions

Consider the dynamic system (10) with conditionally optimal entropy operator (5) and (11).

Theorem 4. *Let the following conditions hold*:

(a) *the conditions of Theorem 1*;
(b) *nonempty compact subsets $X_i^* \subset R_+^n$ $(i = 1, \dots, n)$ exist such that for $x \in X_i^*$,*

$$a_i - (P^i y(x))x_i \geqslant 0, \tag{15}$$

and for $x \in \overline{X}_i^*$,

$$a_i - (P^i y(x))x_i < 0, \tag{16}$$

where $\overline{X}_i^*$ is the complement of the subset X_i^* and P^i is the ith-row of the matrix P.

Then there exists in R_+^n a bounded set $Y \supset (\bigcup X_i^*)$ such that for all initial points $x(0) \in R_+^n$ trajectories $x(t) \in Y$ for $t \geqslant t(x(0)) > 0$.

Proof. (1) Consider the initial point $x(0) \in \mathrm{int} X_i^*$. As the set X_i^* is compact then there exists a neighbourhood of the point $x(0)$ where the first of the conditions (b) is valid for i and possibly for some $i_1, \dots, i_k$. Note that for the numbers $j \neq i, i_1, \dots, i_k$ the second condition (b) will be valid.

In this case the right sides of the $(i, i_1, \dots, i_k)$th equations in (10) are positive, i.e. $\dot{x}_i(0), \dot{x}_{i_1}(0), \dots, \dot{x}_{i_k}(0) > 0$. Hence, the components $x_i(t), x_{i_1}(t), \dots, x_{i_k}(t)$, that arose from the point $x(0)$, will increase monotonically over t. They reach the boundaries of the sets $X_i^*, X_{i_1}^*, \dots, X_{i_k}^*$ at the moments of time $t_i, t_{i_1}, \dots, t_{i_k} > 0$.

For the next moment of time there are two possibilities for the components change over time. One of them is a motion along the boundaries, as $\dot{x}_i(t_i), \dot{x}_{i_1}(t_{i_1}), \dots, \dot{x}_{i_k}(t_{i_k}) = 0$ in the different parts of the boundaries.

Hence in this case *the components $x_i(t), x_{i_1}(t), \dots, x_{i_k}(t)$ are bounded.*

The second possibility for the components change is a motion to the subsets $\overline{X}_i^*, \overline{X}_{i_k}^*, \dots, \overline{X}_{i_k}^*$ where the second conditions (b) are valid. In this case the derivatives of these components become negative and the components $x_i(t), x_{i_1}(t), \dots, x_{i_k}(t)$ decrease for $t > t_i, t > t_{i_1}, \dots, t > t_{i_k}$ respectively. Hence in this case *the components $x_i(t), x_{i_1}(t), \dots, x_{i_k}(t)$ are bounded* too.

Now consider the behaviour of the components for which the second conditions (b) are valid if the initial point belongs to the set X_i^*. As the derivatives of these components are negative, then $x_j(t) \leqslant x_j(0), j \neq i, i_1, \dots, i_k$ for the time interval $0 \leqslant t \leqslant \tau$. For $t > \tau$ there are two possibilities: to continue motion in the sets $\overline{X}_j^*$ or to reach the sets X_j^*. In the first decreasing of the components is continued, and in the second we have the case considered above.

As the trajectory $x(t)$ can visit all the sets X_i^*, $i = 1, \dots, n$, and its boundedness on the R_+^n is guaranteed by the second conditions (b) then the bounded set Y declared in the theorem exist.

(2) Consider the initial point $x(0) \in \overline{X}_i^*$. In this case we have the symmetric picture with the same results. □

5. Labour market model

Labour is one of the components of any production process. However, a labour requirement is finite and its amount (the so-called labour demand) is determined by profitability of production. All able to work population (belonging to the age-window of ability to work) is a labour supply in general. Labour demand and supply have met in the labour market and the latter forms the age-specific structures of employment and unemployment.

The main principle of the labour market is a competition for jobs among participants. Certainly, it is not possible to model a competition on the microlevel, i.e. among people. Usually all participants are divided into groups and the competition among these groups is considered.

We will consider groups created by date of birth (date of birth cohorts $\tilde{c}$). Such classification is very convenient as it gives a possibility to transform the cohort-groups into the age-groups and to the age-specific structure of the employment.

The process of time evolution of cohort- or age-structures of the employment is quasi-deterministic and slow enough (van Imhoff and Henkens, 1998). A competition between cohorts is a stochastic process which is realized significantly faster than change of the cohort-structure of employment.

5.1. Equations of the model

The state of the labour market at the calendar time $\tilde{t}$ is characterized by a cohort-specific employment structure (CSE). The CSE is a distribution of the relative number of employed workers over cohorts $\tilde{c}$ and time $\tilde{t}$, which is given by the density function denoted as $w(\tilde{c},\tilde{t})$. In order to study the labour market dynamics we introduce the notion of a *rate of employment change (REC)*, or *CSE-rate (CSE-R)* as a relative temporary rate of the CSE function, denoted by $\gamma(\tilde{c},\tilde{t})$:

$$\frac{1}{w(\tilde{c},\tilde{t})}\frac{\mathrm{d}w(\tilde{c},\tilde{t})}{\mathrm{d}\tilde{t}} = \gamma(\tilde{c},\tilde{t}), \tag{17}$$

where $\tilde{c}$ is fixed.

We will consider the population with working ages $\tilde{a}$ from interval $\widetilde{A}_w = [\tilde{a}_0,\tilde{a}_1]$ (age-window $\widetilde{A}_w$). Dynamics of the labour market will be considered within the time interval $\widetilde{T} = [\tilde{t}_0,\tilde{t}_1]$. Without loss of generality we consider $\tilde{t}_1 = \tilde{t}_0 + \tilde{a}_0$.

Introduce the following shifted variables: $a = \tilde{a} - \tilde{a}_0$, $a \in A_w = [0,a^*]$; $t = \tilde{t} - \tilde{t}_0$, $t \in T = [0,\tilde{a}_0]$; $c = \tilde{c} - \tilde{c}_0$, $c \in K = [0,\tilde{a}_1]$. In these expressions $a^* = \tilde{a}_1 - \tilde{a}_0$.

There is one-to-one relation among cohort, age and time:

$$c = t - a + a^*. \tag{18}$$

At each moment of time t only part of cohorts c in the interval K belongs to the age-window A_w. All such cohorts belong to the subset

$$C_t = [t, t + a^*] \in K, \quad t \in T. \tag{19}$$

To describe the labour market state at time t we use three density functions which are closely connected to each other:

- cohort-specific employment structure (CSE):

$$w(c,t) = \chi(t-c,t), \quad c \in C_t, \ t \in T, \tag{20}$$

- age-specific employment structure (ASE):

$$\chi(a,t) = w(t-a,t), \quad a \in A_w, \ t \in T, \tag{21}$$

- distribution of employed cohorts (DEC):

$$k(c,t) = \begin{cases} w(c,t) & \text{for } c \in [0, \tilde{a}_0], \quad t \in [0, c], \\ 0 & \text{for } c \in [0, \tilde{a}_0], \quad t \in [c+1, \tilde{a}_0], \\ w(c,t) & \text{for } c \in [\tilde{a}_0 + 1, a^*], \quad t \in [0, \tilde{a}_0], \\ 0 & \text{for } c \in [a^* + 1, \tilde{a}_1], \quad t \in [0, c - (a^* + 1)], \\ w(c,t) & \text{for } c \in [a^* + 1, \tilde{a}_1], \quad t \in [c - a^*, \tilde{a}_0]. \end{cases} \tag{22}$$

For modelling of the labour market dynamics we use the REC-function $\gamma(c,t)$ (17). Then we obtain the following differential equations:

$$\frac{\mathrm{d}w(c,t)}{\mathrm{d}t} = w(c,t)\gamma(c,t), \quad c \in C_t, \ t \in T, \tag{23}$$

where C_t is determined by (19).

The simple Euler scheme with step $h = 1[year]$ gives the following difference equations, which describe the model of the labour market:

$$w(c,t+1) = w(c,t)[1 + \gamma(c,t)], \quad c \in C_t, \ t \in T_1, \tag{24}$$

where C_t is defined in the expression (19) and

$$T_1 = T \setminus \tilde{a}_0 = [0, \tilde{a}_0 - 1]. \tag{25}$$

The initial distribution $w(c,0)$ is given by the following conditions:

$$\begin{aligned} & w(c,0) = w^0(c), \quad 0 < w^0 < 1, \ c \in C_0, \\ & \sum_{c \in C_0} w^0(c) = 1. \end{aligned} \tag{26}$$

The boundary conditions $w^{\mathrm{B}}(t+1)$ take the form:

$$w(t + a^* + 1, t+1) = w^{\mathrm{B}}(t+1), \quad 0 < w^{\mathrm{B}}(t+1) < 1, \ t \in T_1. \tag{27}$$

Since initial and boundary conditions are positive, all solutions of the differential equations (23) are positive too. Hence the system (23) is positive dynamic system.

However, in difference equations (24) negative solutions are possible. Also the distribution $w(c, t+1)$ (24) is not a density function.

Summing up, the *main equations of the labour market model* can be represented in the following form:

$$\hat{w}(c,t+1) = \begin{cases} w(c,t)[1 + \gamma(c,t)], \quad c \in C_t & \text{if } \hat{w}(c,t+1) \geqslant 0, \\ 0 & \text{if } \hat{w}(c,t+1) < 0, \end{cases} \tag{28}$$

$$w(c,t+1) = \frac{\hat{w}(c,t+1)}{N(t+1)}, \quad c \in (C_t \setminus t), \tag{29}$$

where

$$N(t+1) = \sum_{c \in (C_t \setminus t)} \frac{\hat{w}(c,t+1)}{1 - w^{\mathrm{B}}(t+1)}.$$

5.2. REC-function

We take into account cohorts competition and labour supply–demand interaction which are the main causes of changes in the state of the labour market.

Competing for jobs cohort c wants *to keep* its state *in* the labour market and cohorts $l \neq c$ want *to push c from* the labour market. Supply–demand interaction characterizes an influence socioeconomic system on the labour market. It is an aggregate factor of the economic level, population activity, level of self-reproduction, life style and so on.

According to this phenomenology we consider three components of the REC-function:

- $\rho(c,t)$—own competitiveness of the cohort c;
- $\kappa(c,t)$—comparative competitiveness of the cohorts c and l;
- $\sigma(c,t)$—labour supply–demand interaction.

Represent the REC-function $\gamma(c,t)$ in the following form:

$$\gamma(c,t) = \rho(c,t) + \kappa(c,t) + \sigma(c,t). \tag{30}$$

The main factor of own competitiveness is a cohort(age)-factor (Klevmarken, 1993). Age is linked with an employment protection and cohort's skills. This phenomenon can be reflected by the following equation:

$$\rho(c,t) = \rho(c) = \rho \exp(-\zeta c), \quad c \in C_t, \tag{31}$$

where ρ and ζ are parameters.

Comparative competitiveness is the sum of the comparative competitiveness of the cohorts $l \in C_t \setminus c$ in relation to cohort c. The latter is proportional to $w(l,t)$ and the coefficient of proportion depends on the utility of cohorts l and c.

According to the utility function approach comparative competitiveness depends on cohort-distance factors $\Gamma(c,l)$ and comparative utility factors $\Theta(c,l,t)$. Since these factors are independent, we can describe their (Θ and Γ) influence multiplicatively.

We use an exponential function between the distance- and the utility-factors:

$$\Gamma(c,l) = \exp(-\alpha|c-l|), \tag{32}$$

$$\Theta(c,l,t) = \theta \exp[\eta(u(c,t) - u(l,t))], \tag{33}$$

where α, η, θ are parameters; $u(c,t)$ is the utility function of the cth cohort. The main factor influencing the cohort utility is the number of employed persons in the cohort $x(c,t)$.

We assume a decreasing marginal gain in utility for every additional employee of the cohort, which can be represented by the logarithmic function

$$u(c,t) = \ln x(c,t). \tag{34}$$

We restore the distribution of the employed persons over cohorts using the macrosystem technique (Popkov, 1995), in which jobs at time t are allocated randomly over cohorts c, according to the prior probabilities $w(c,t)$, while taking into account the constraints with respect to the total labour demand and the cohort-specific labour supply. In this case the distribution $x(c,t)$ can be determined by the entropy maximization problem

$$\begin{gathered} H[X(t)] \Rightarrow \max_x, \\ \sum_{c \in C_t} x(c,t) = R^{\mathrm{E}}(t), \\ 0 < x(c,t) < S(c,t), \quad c \in C_t, \end{gathered} \tag{35}$$

where the entropy $H[X(t)]$ is determined by the following expression:

$$H[X(t)] = -\sum_{c\in C_t}\left(x(c,t)\ln\frac{x(c,t)}{w(c,t)} + (S(c,t)-x(c,t))\ln(S(c,t)-x(c,t))\right), \tag{36}$$

and $X(t) = \{x(t,t),\dots,x(t+a^*,t)\}$.

This problem describes the mapping of the functions $w(c,t)$, $S(c,t)$, $R^{\mathrm{E}}(t)$ into the function $x(c,t)$. The operator realizing this mapping is the *entropy operator*.

By combining the expressions (32) and (33) we arrive at the formula for the comparative competitiveness:

$$\kappa(c,t) = \theta\sum_{l\in C_t\setminus c}\exp(-\alpha|c-l|)\left(\frac{x(c,t)}{x(l,t)}\right)^{\eta}. \tag{37}$$

Consider the third component of the REC-function (30) and introduce the following relative variables:

$$r^{\mathrm{E}}(t) = \frac{R^{\mathrm{E}}(t)}{S(t)},\quad l(c,t) = \frac{S(c,t)}{S(t)}. \tag{38}$$

In the general case the demand–supply component $\sigma(c,t)$ (30) is a function of both $r^{\mathrm{E}}(t)$ and $l(c,t)$. We use a bilinear function:

$$\sigma[r^{\mathrm{E}}(t), l(c,t)] = \beta r^{\mathrm{E}}(t)l(c,t), \tag{39}$$

where β is a parameter.

In the expression (37) $x^*(c,t)$, $c\in C_t$ is an entropy optimal distribution of employed persons over cohorts which is determined by the solution of the conditional maximization of the entropy (35), (36) (Popkov, 1995):

$$x^*(c,t) = \frac{w(c,t)S(c,t)}{w(c,t)+z^*(t)[1-w(c,t)]}, \tag{40}$$

where Lagrange exponential multiplier $z^*(t)$ is determined by the solution of the equation

$$\psi(z,t) = \frac{1}{R^{\mathrm{E}}(t)}\sum_{c\in C_t}\frac{w(c,t)S(c,t)}{w(c,t)+z^*(t)[1-w(c,t)]} = 1. \tag{41}$$

Thus, Eqs. (28)–(31), (37), (39)–(41) describe the dynamic model of the labour market, which belongs to the positive dynamic system with the entropy operator.

5.3. Parameter identification

We can see from the equations of the model that they have certain numbers of unknown parameters: $y_0=\rho$, $y_1=\zeta$, $y_2=\eta$, $y_3=\theta$, $y_4=\alpha$, $y_5=\beta$.

For identification of these parameters we use the following real data in the time interval T: $\chi_r(a,t)$, $S_r(a,t)$, $R_r^{\mathrm{E}}(t)$. Using the data of the ASE-function $\chi_r(a,t)$ it is possible to calculate the real CSE-function $w_r(c,t)$ and the real DEC-function $k_r(c,t)$ (see the formulas (20)–(22)).

We have the real ASE- and DEC-functions $\chi_r(a,t)$, $k_r(c,t)$ and the modelled functions $\chi(a,t,y)$, $k(c,t,y)$. It is necessary to define a criterion of their closeness.

Define the maximal row-error for the ASE-functions in the following form:

$$\varepsilon_t(y) = \max_{t\in T_1}\varepsilon(y,t), \tag{42}$$

where

$$\varepsilon(y,t)=\frac{\sum_{a\in A_w}(\chi(a,t)-\chi_r(y,t))^2}{\sum_{a\in A_w}\chi^2(a,t)+\sum_{a\in A_w}\chi_r^2(a,t)}. \tag{43}$$

The divergence between the DEC-functions is estimated by the following maximal column-error:

$$\varepsilon_c(y)=\max_{c\in K}\varepsilon(y,c), \tag{44}$$

where

$$\varepsilon(y,c)=\frac{\sum_{t\in T_1}(k(c,t)-k_r(c,t))^2}{\sum_{t\in T_1}k^2(c,t)+\sum_{t\in T_1}k_r^2(c,t)}. \tag{45}$$

Also the entropy criterion is introduced for the ASE-functions:

$$H(y)=\sum_{a\in A_w}\sum_{t\in T_1}\chi(a,t)\ln\frac{\chi(a,t)}{\chi_r(a,t)}, \tag{46}$$

which is used for regularization of the identification problem.

Finally, the identification criterion can be represented in the following form:

$$\varepsilon(y)=v\varepsilon_t(y)+(1-v)\varepsilon_c(y)+\lambda H(y), \tag{47}$$

where the weight coefficients are $v\in[0,1]$ and $\lambda\in[0,1]$.

The optimal parameters of the model are

$$y^*=\arg\min\varepsilon(y). \tag{48}$$

It is known that the function $\varepsilon(y)$ has many minimum points. In our case we have only information about values of the identification criterion $\varepsilon(y)$ and the domains of the parameters localization $y_i^-\leqslant y_i\leqslant y_i^+$, $i\in[0,5]$.

For search of the quasi-global minimum we apply combination of the local gradient search and "intelligent" random jumps (with analysis of the results of certain numbers of random jumps). If during some random jumps the value of the function $\varepsilon(y)$ is not decreased then some parameters of jumps are changed. In our case the direction and values of the random jumps are changed.

The identification algorithm can be represented in the following form:

Step 0. Initial conditions. Input initial values of the parameters $y=y^0$ and calculate the function $\varepsilon(y^0)$, and memorize of the values y_i^0, $i\in[0,5]$ and $\varepsilon(y^0)$.

Step 00. Local gradient search; determination of the gradient components:

$$\nabla_y(\varepsilon(y^0))=\frac{1}{\Delta}[\varepsilon(y^0+\Delta)-\varepsilon(y^0)].$$

Step 01. Gradient step:

$$y^{01}=y^0-p\nabla_y(\varepsilon(y^0));$$

calculate $\epsilon(y^{01})$ and check the condition

$$\text{if }|\varepsilon(y^0)-\varepsilon(y^{01})|\leqslant\delta,\quad\text{then } stop.$$

Step 0s. The sth step is stop; comparison $\varepsilon(y^{0s})$ with $\varepsilon(y^0)$ and memorizing of the smallest (for instance, $\varepsilon(y^{0s})$) and y^{0s}.

Step 1. Random jump.

$$y^1 = y^{0s} + \vartheta^1,$$

and calculation of the value $\varepsilon(y^1)$, and comparison it with $\varepsilon(y^{0s})$.

Steps 10–1s. Local gradient search.

Step 2. Random jump; comparison $\epsilon(y^{1s})$ with $\min(\epsilon(y^1)), \varepsilon(y^{0s})$ and memorizing the smallest from them.

Step N. Random jump; N is the fixed input variable.

The parameters of the local gradient search: test step Δ, gradient step p and error δ are input parameters of the algorithm.

5.4. Model test

The model was tested on the problem of the parameters identification for nine EU-countries (Belgium, Denmark, France, Greece, Ireland, Italy, Luxemburg, the Netherlands, United Kingdom). We use the data base on the labour force from 1983 till 1996, presented by the Netherlands Interdisciplinary Demographic Institute (NIDI).

Identification of the model parameters has two aims. One of them is a pragmatic aim: if we know the model parameters then we can use this model for prediction of the labour market evolution. And the other is a cognitive aim: could we take six parameters of the model as an image of the country labour market? If yes, then the set of six parameters reflects (certainly, in some framework) the employment behaviour in the country.

Results of identification are given in Table 1. In Table 1 and the other tables the following notations are introduced: BG—Belgium, DN—Denmark, F—France, GR—Greece, IR—Ireland, I—Italy, LB—Luxemburg, NL—the Netherlands, UK—United Kingdom.

The DEC-functions $k(c, 1989)$ (real and modelled) for France and the Netherlands are shown in Figs. 1 and 2.

Now we attempt to classify the countries over the characteristics of the labour market.

Consider the six-dimensional space of the parameters and define a distance between the vectors (or points) y_i and y_j in the following form:

$$d(y^i, y^j) = \frac{\sum_{s=0}^{5}(y_s^i - y_s^j)^2}{\sum_{s=0}^{5}[y_s^i]^2 + \sum_{s=0}^{5}[y_s^j]^2}. \tag{49}$$

It is a positive function which is equal to zero when the points i and j coincide. The values of this function are shown in Table 2.

Table 1

Country	ρ	ζ	η	θ	α	β
BG	29.230	0.010	0.001	0.001	5.627	37.519
DN	45.004	0.010	0.002	0.001	2.000	82.000
F	40.002	0.010	0.002	0.001	2.000	90.000
GR	9.301	0.010	0.001	0.001	0.502	0.001
IR	11.684	0.010	0.002	0.001	2.000	50.008
I	20.002	0.010	0.002	0.002	2.000	100.00
LB	40.004	0.010	0.002	0.001	2.006	100.00
NL	48.549	0.010	0.002	0.002	2.254	99.975
UK	54.972	0.010	0.004	0.001	1.527	90.049

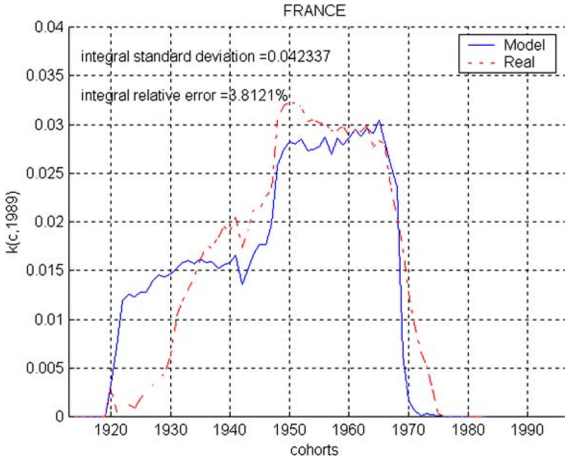

Fig. 1.

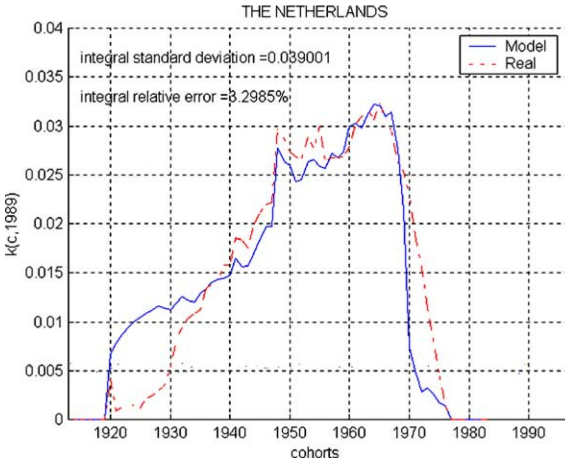

Fig. 2.

Using this table we can choose a group of countries, a distance between which is less than some threshold δ. The results of such a classification are given in Table 3.

Now we compare this classification with an other which we can obtain from comparison of the real age-specific employment structures of the EU-countries.

We will form the criterion of classification which could take into account the integral distinction between the age-specific employment structures and the activity rates of young people. The first is measured as the normalized quadratic error between two functions for fixed time t:

$$\varrho^{i,j}(t) = \frac{\sum_{a=0}^{a^*}[\chi_r^i(a,t) - \chi_r^j(a,t)]^2}{\sum_{a=0}^{a^*}[\chi_r^i(a,t)]^2 + \sum_{a=0}^{a^*}[\chi_r^j(a,t)]^2}. \tag{50}$$

We consider time $t = 4$ (1989). The values of this function are represented in Table 4.

Table 2

Country	BG	DN	F	GR	IR	I	LB	NL	UK
BG	0.000	0.191	0.724	0.823	0.141	0.690	0.607	0.581	0.451
DN	0.191	0.000	0.205	0.860	0.165	0.271	0.191	0.171	0.112
F	0.724	0.205	0.000	0.964	0.177	0.023	0.004	0.006	0.007
GR	0.823	0.860	0.964	0.000	0.950	1.000	0.969	0.959	0.937
IR	0.141	0.165	0.177	0.950	0.000	0.157	0.179	0.194	0.182
I	0.690	0.271	0.023	1.000	0.158	0.000	0.012	0.216	0.034
LB	0.607	0.191	0.004	0.969	0.180	0.012	0.000	0.002	0.007
NL	0.581	0.171	0.006	0.960	0.194	0.021	0.002	0.000	0.003
UK	0.451	0.112	0.008	0.937	0.182	0.034	0.007	0.003	0.000

Table 3

δ	Group of countries
0.002	LB, NL
0.006	F, LB, NL, UK
0.02	F, I, LB, NL, UK
0.05	F, I, LB, NL, UK

Table 4

Country	BG	DN	F	GR	IR	I	LB	NL	UK
BG	0.000	0.694	0.109	0.719	0.619	0.380	0.157	0.287	0.760
DN	0.693	0.000	0.475	0.569	0.388	0.339	0.397	0.276	0.118
F	0.109	0.475	0.000	0.393	0.616	0.142	0.178	0.254	0.516
GR	0.718	0.569	0.393	0.000	1.000	0.138	0.708	0.811	0.615
IR	0.619	0.388	0.616	1.000	0.000	0.691	0.263	0.249	0.305
I	0.380	0.339	0.142	0.138	0.691	0.000	0.342	0.392	0.395
LB	0.157	0.397	0.178	0.707	0.263	0.342	0.000	0.108	0.360
NL	0.287	0.276	0.254	0.811	0.249	0.392	0.108	0.000	0.264
UK	0.760	0.118	0.516	0.615	0.305	0.395	0.360	0.264	0.000

The activity rates $A^i(t)$ of young people in the labour market are determined by the ASE-function $\chi^i_r(a,t)$ in the following form

$$A^i(4) = \sum_{a=0}^{14} \chi^i_r(a,4). \tag{51}$$

The values of the activity rate are represented in Table 5.

It is plausible to assume that the higher the general activity rate in both countries i and j the less weight of the integral error $\varrho^{i,j}$. So the generalized criterion of classification takes the form

$$J^{i,j}(4) = \frac{\varrho^{i,j}(4)}{A^i(4) + A^j(4)}. \tag{52}$$

The values of this criterion are represented in Table 6.

From this table we can see that the minimal value of the generalized error $J^{ij}(4)$ is equal to 0.063. This generalized error corresponds to Luxemburg (LB, 7) and the Netherlands (NL, 8). Hence, they create the class of the countries closest to each other from the point of view of the employed people behaviour in the labour market.

Table 5

Country	BG	DN	F	GR	IR	I	LB	NL	UK
$A^i(4)$	0.172	0.226	0.163	0.126	0.256	0.154	0.218	0.237	0.239

Table 6

Country	BG	DN	F	GR	IR	I	LB	NL	UK
BG	0.000	0.541	0.118	1.000	0.425	0.434	0.127	0.213	0.560
DN	0.541	0.000	0.390	0.603	0.203	0.295	0.244	0.156	0.067
F	0.118	0.390	0.000	0.576	0.446	0.170	0.151	0.198	0.401
GR	1.000	0.603	0.576	0.000	0.935	0.215	0.776	0.818	0.617
IR	0.425	0.203	0.446	0.934	0.000	0.529	0.142	0.124	0.151
I	0.434	0.295	0.170	0.215	0.529	0.000	0.307	0.324	0.325
LB	0.127	0.244	0.151	0.776	0.142	0.307	0.000	0.063	0.209
NL	0.213	0.156	0.198	0.818	0.124	0.324	0.063	0.000	0.141
UK	0.560	0.067	0.401	0.617	0.151	0.325	0.209	0.141	0.000

Consider Table 4, which contains the values of the normalized quadratic error $\varrho^{ij}(4)$ (50). This variable is a part of the generalized criterion $J^{ij}(4)$ (52) (without the activity rates). From Table 4 we can see that the minimal value of $\varrho^{ij}(4)$ is equal to 0.108. This error corresponds to the same class (LB, NL).

Hence, we can consider the class (LB, NL) as stable enough: classification for two criteria (ϱ and J) gives the same result.

If we compare this class with the class created by the closeness of the model parameters (see Table 3, for $\delta = 0.002$) then we can see that they coincide and the model parameters represent the image of the country labour market.

Thus, the concept of the positive dynamic system with entropy operator is useful for modelling of labour market dynamics. A model of the employment structures in terms of cohorts for nine EU countries was constructed, its parameters were identified and its adequacy was tasted.

Appendix A

Proof of Theorem 1. Consider the quadratic form

$$G(y,x,\lambda) = \langle L_{yy}(y,x,\lambda)h, h\rangle, \tag{A.1}$$

where

$$L_{yy}(y,x,\lambda) = \left[\frac{\partial^2 L}{\partial y_j \partial y_i}\right] = \begin{cases} \frac{1}{y_j} - \sum_{k=1}^{r} \lambda_k \frac{\partial^2 \phi_k(y,x)}{\partial y_j^2} & \text{for } j = i, \\ -\sum_{k=1}^{r} \lambda_k \frac{\partial^2 \phi_k(y,x)}{\partial y_j \partial y_i} & \text{for } j \neq i. \end{cases} \tag{A.2}$$

According to (A.2) the quadratic form (A.1) takes the form

$$G = -\sum_{j=1}^{m} \frac{1}{y_j} h_j^2 - \sum_{k=1}^{r} \lambda_k \left(\sum_{i,j=1}^{m} \frac{\partial^2 \phi_k}{\partial y_j \partial y_i} h_j h_i \right). \tag{A.3}$$

The first component in this equality is negative and it is equal to zero if $h \equiv 0$.

Show that for the BE-operator the Lagrange multipliers $\lambda_k \geqslant 0$, $k = 1, \dots, r$, if the conditions of Theorem 1 are valid.

From (13) we have

$$y_j - \tilde{w}_j(x)\exp\left(-\sum_{k=1}^{r}\lambda_k\beta_{kj}(y,x)\right) = 0, \quad j = 1,\dots,m,$$

where

$$\beta_{kj}(y,x) = \frac{\partial\phi_k}{\partial y_j}.$$

Introduce the variables

$$z_k = \exp(-\lambda_k) \geqslant 0.$$

Then the first equations in (13) can be represented in the following form:

$$\psi_j(y,x,z) = y_j - \tilde{w}_j(x)\prod_{k=1}^{r} z_k^{\beta_{kj}(y,x)} = 0, \quad j = 1,\dots,m. \tag{A.4}$$

Consider the vector $z \geqslant 0$ and direction $p = \{p_k \geqslant 0,\ k = 1,\dots,r\}$. Determine the derivative of the function Ψ_j at the point z with respect to direction p. We have

$$\left.\frac{d\psi_j}{d\delta}\right|_{\delta=0} = -\tilde{w}_j(x)\sum_{l=1}^{r}\prod_{k=1,k\neq l}^{r}(z_k+\delta p_k)^{\beta_{kj}(y,x)} \times \beta_{lj}(y,x)p_l(z_l+\delta p_l)^{\beta_{lj}(y,x)-1}.$$

It is seen that this derivation is negative for any $z, p \geqslant 0$, if the condition (a) of Theorem 1 is valid. Then the functions $\psi_j(y,x,\lambda)$ are strong monotonically decreasing over y.

Now consider the condition (d) of Theorem 1. From this condition it is followed that

$$y \in Y = \{0 < y_j \leqslant \tilde{w}_j(x),\ j = 1,\dots,m\}. \tag{A.5}$$

Consider the set $Z = \{z : 0 \leqslant z_k \leqslant 1,\ k = 1,\dots,r\}$. We can see from (A.4) that

$$\psi_j(y,x;z_1,\dots,z_{h-1},0,z_{h+1},\dots,z_r) > 0,$$

$$\psi_j(y,x;1) \leqslant 0, \quad j = 1,\dots,m.$$

Taking into account that the functions Ψ decrease strong monotonically we have that in the set Z there exists at least one solution z^*, i.e. $0 < z_k^* \leqslant 1$, $k = 1,\dots,r$. According to (A.4) for z^* there exists the vector y^* belonging to the set Y.

Thus, we have that $0 \leqslant \lambda_k < \infty$, $k = 1,\dots,r$, and the quadratic form (A.4) is strong negative defined. □

Consider the Jacobian of the system (13):

$$J(y,x,\lambda) = \begin{bmatrix} A(y,x,\lambda) & B'(y,x,\lambda) \\ B(y,x,\lambda) & 0 \end{bmatrix}, \tag{A.6}$$

where

$$A(y,x,\lambda) = \left[\frac{\partial^2 L}{\partial y_j \partial y_i}\right], \tag{A.7}$$

$$B(y,x,\lambda) = \left[-\frac{\partial\phi_k(y,x)}{\partial y_j}\right]. \tag{A.8}$$

Lemma A.1. *Let* $(m \times m)$*-matrix A be a definite positive* (*negative*) *one and* $(r \times m)$*-matrix B has the full rank r. Then* $\det J \neq 0$.

Proof. To prove the lemma, let us assume the opposite, i.e. let $\det J = 0$. Then there exists an eigenvector $z = \begin{pmatrix} u \\ v \end{pmatrix}$ which corresponds to the zero eigenvalue and is such that

$$Jz = 0. \tag{A.9}$$

Taking into account the structure of J (A.6), we obtain

$$Au + B'v = 0, \quad Bu = 0. \tag{A.10}$$

By multiplying the first equality by u' from left, we obtain

$$u'Au + v'Bu = 0.$$

But from the second equality in (A.10) we have $v'Bu = 0$ and $u'Au = 0$. Since the matrix A is defined as positive (negative) one, the corresponding quadratic form is zero only on zero vectors ($u \equiv 0$). Return to the first equality in (A.10). It implies that $Bv = 0$ for $u = 0$. As B has the full rank equal r then $v \equiv 0$. Thus the eigenvector $z \equiv 0$. □

The proofs of Theorems 2 and 3 follow from the global theorem of implicit function existence and local theorem of implicit function differentiability. The latter is well known, so we give only the first.

Consider the equation

$$W(u,x) = 0, \tag{A.11}$$

where $u = \{y, \lambda\} \in R^{n+r}$ and the $(m + r)$ vector-function

$$W(u,x) = \begin{bmatrix} \Psi(u,x) \\ \Phi(u,x) \end{bmatrix}. \tag{A.12}$$

Theorem A.1. *Let the function* $W(u,x)$ ((A.11) *and* (A.12)) *be continuous with respect to its variables and let the following conditions hold for any fixed* $x \in R^n$:

(a) $\det J(u,x) \neq 0$ *for any* $u \in R^{m+r}$;
(b) $\lim_{\|u\| \to \infty} W(u,x) = \pm\infty$.

Then there exists the unique implicit function $u(x)$ *defined on* R^n.

Proof. The function $W(u,x)$ generates the vector field

$$F_x(u) = W(u,x) \tag{A.13}$$

for any fixed $x \in R^n$. The field is continuous by the theorem conditions.

Introduce the vector field

$$\Pi_v(u) = F_x(u) - v, \tag{A.14}$$

where $v \in R^n$ is a fixed vector.

It is clear that, according to (b), for the fixed v the vector field $\Pi_v(u)$ has no zeros on the spheres $\|u\| = \rho$ if ρ is large enough. Therefore the rotation of $\Pi_v(u)$ is defined on the spheres $\|u\| = \rho$ with sufficiently large ρ (Krasnoselsky and Zabreiko, 1975).

Consider two vector fields produced in (A.14) by the vector v, namely:

$$\Pi_{v_1}(u) = F_x(u) - v_1, \qquad \Pi_{v_2}(u) = F_x(u) - v_2. \tag{A.15}$$

The vector fields are homotopic on spheres with sufficiently large radii and so they have the same rotations:

$$\gamma(\Pi_{v_1}) = \gamma(\Pi_{v_2}). \tag{A.16}$$

The vector fields Π_{v_1} and Π_{v_2} do not degenerate on the spheres with large radii, but each of them can have a number of singular points in the ball $\|u\| \leqslant \rho_1 < \rho$. Denote $\kappa(v_1)$ and $\kappa(v_2)$ as the numbers of singular points for the fields Π_{v_1} and Π_{v_2}, respectively. Since these fields are homotopic, we have

$$\kappa(v_1) = \kappa(v_2) = \kappa. \tag{A.17}$$

Let the field $\Pi_v(u)$ have κ singular points in the ball $\|u\| \leqslant \rho_1 < \rho$. These points are isolated by the condition (a) of the theorem.

According to Krasnoselsky and Zabreiko (1975) the index of the singular point u^0 is

$$\mathrm{ind}(u^0) = (-1)^{\beta(u^0)}, \tag{A.18}$$

where $\beta(u^0)$ is the number of eigenvalues of the Jacobian $J(u^0, x)$ which have negative real parts. The definition shows that the index value, namely +1 or −1, depends on the evenness of $\beta(u^0)$, not on its absolute value.

The evenness of $\beta(u^0)$ turns out to be the same for all the singular points. This follows from condition (a) of the theorem. In fact, since $\det J(u,x) \neq 0$ for any $x \in R^n$, the eigenvalues of $J(u,x)$ can pass from the left half-plane to the right one only by pairs, i.e. real eigenvalues are transformed into pairs of complex-conjugates and the latter then intersect the imaginary axis.

By taking into account this fact and (A.17) and (A.18), we obtain the rotation of homotopic field (A.14):

$$\gamma(\Pi_v) = \kappa(-1)^{\beta}, \tag{A.19}$$

where β is the number of eigenvalues of the matrix $\Pi'_v(u) = J(u,x)$ which have a negative real part for some u.

Now let us show that the vector field $\Pi_v(u)$ has a unique singular point in the ball $\|u\| \leqslant \rho_1 < \rho$. Consider the equation:

$$\Pi_v(u) = F_x(u) - v = 0. \tag{A.20}$$

Let the equation have κ singular points for any fixed v, i.e. there exist κ functions $u_1(v), \ldots, u_\kappa(v)$. So the equation (A.20) determines a multivalued function $u(v)$, with κ its branches being isolated (the latter follows from isolation of the singular points). Each branch $u_i(v)$ determines in R^n an open subset U_i (by condition (b) of the theorem), with

$$\bigcup_{i=1}^{\kappa} U_i = R^{m+r}.$$

This is possible only if $\kappa = 1$. Hence, the rotation of $\Pi_v(u)$ equals $(-1)^\beta$ and, by the homotopy, the rotation of $\Pi_0(u) = F_x(u)$ equals $(-1)^\beta$ too.

Hence, for any $x \in R^n$ there exists the unique function $u(x)$ for which the function $W(u,x)$ is zero. □

Proof of Lemma 1. See (A.5). The FE-operator is limited by $\max(S_1, \ldots, S_m)$ due to the form of the entropy function (13). □

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